10

Matrices

10.1. Matrix

Definition. An arrangement of mn numbers belonging to a number system F (real or complex) into m rows and n columns is called a *matrix of order $m \times n$ over F*. (*K.U. 1981*)

For example :

(*i*) $\begin{bmatrix} 2 & -3 & i \\ 3 & 8 & 6+2i \end{bmatrix}$ is a matrix of order 2×3,

as it has two rows and three columns.

(*ii*) $\begin{bmatrix} 1 & 8 & -7 \\ 2 & 5 & 6 \\ i+2 & 0 & 4 \end{bmatrix}$ is a matrix of order 3×3.

(*iii*) In general a matrix of order $m \times n$ can be written as

$$\begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{i1} & a_{i2} \ldots\ldots a_{in} \\ \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} \end{bmatrix}$$

which can be briefly written as $[a_{ij}]_{m \times n}$.

Note 1. We shall denote a matrix by capital latters, A, B, C ...... etc.

2. The element a_{ij} is that which occurs in the ith row and jth col. The first suffix indicates row number, while the second suffix indicates the col. number.

3. Members of the number system F are called scalars relative to the matrix.

4. The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots, a_{nn}$ in which both suffixes are same, are called the **diagonal elements,** all other are called non-diagonal elements.

Thus a_{ij} is a diagonal element if $i = j$

and a_{ij} is non-diagonal eletments if $i \neq j$.

5. The line along which the diagonal elements.

$a_{11}, a_{22}, \ldots\ldots, a_{nn}$ lie is called the **Principal Diagonal.**

10.2. Different Types of Matrices

1. Zero Matrix or Null Matrix. A matrix each of whose element is zero is called a zero matrix or null matrix.

e.g., $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are zero matrices respectively of order 2×3 ; 3×2 and 3×3.

In general, a *zero* matrix of order $m \times n$ is denoted by $O_{m \times n}$.

Note. A matrix which is not a zerŏ-matrix is called a non-zero matrix.

2. Square matrix. A matrix in which the number of rows is equal to the number of columns is called a Square matrix.

A square matrix of order $n \times n$ called square matrix of order n.

A matrix which is not square is called a **rectangular matrix.**

3. Row-matrix or Row-Vector. A matrix of type $1 \times n$ *i.e.*, having only one row is called a **row-matrix.** For example, $[1, -3, -7, i, 0]$ is a row-matrix of order 1×5. (*M.D.U. 1983*)

4. Column-matrix or Column-vector. A matrix of type $m \times 1$ *i.e.*, having only one column is called a *column-matrix*. (*M.D.U. 1983*)

For example, $\begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix}$ is a column matrix of order 3×1.

5. Diagonal Matrix. A square matrix in which all non-diagonal elements are zero is called a *diagonal matrix*.

In symbols. The matrix $A = [a_{ij}]_{n \times n}$ is diagonal matrix if $a_{ij} = 0$ for $i \neq j$. Thus

$\begin{bmatrix} 2 & 0 \\ 0 & -8 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are diagonal matrices.

Note. The diagonal matrix $\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$ can be briefly written as diagonal $[x_1, x_2, x_3]$.

6. Scalar Matrix. A diagonal matrix in which all diagonal elements are equal is called a scalar matrix.

In symbols. The square matrix $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = 0$ $(i \neq j)$ and $a_{ij} = k$ for $i = j$.

e.g. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ is a scalar matrix.

7. Unit Matrix or Identity Matrix. A *scalar* matrix *of order n* in which all diagonal elements are unity is called a **unit or identity matrix** and is generally denoted by I_n. *(M.D.U. 1983)*

In Symbols. A square matrix $A = [a_{ij}]_{n \times n}$ will be a unit or identity matrix if

(*i*) $a_{ij} = 0$ for $i \neq j$ and (*ii*) $a_{ij} = 1$ for $i = j$.

8. Tri-angular Matrix. These are of two types :

(*a*) **Upper-triangular matrix.** It is a matrix in which all elements below the principal diagonal are zero

e.g., $$\begin{bmatrix} 1 & -2 & i \\ 0 & 5 & -7 \\ 0 & 0 & 9 \end{bmatrix}$$

(*b*) **Lower-triangular matrix.** It is a matrix in which all elements above the Principal diagonal are zero

e.g., $$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 7 & 0 \\ 3 & 8 & 4 \end{bmatrix}$$

9. Sub-matrix. A matrix B obtained by deleting some rows or columns or both of a matrix A, is called a sub-matrix of A.

For example, if $A = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 1 & 3 & 9 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, then the matrices

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}, [0, 0, 1, 2] \text{ etc.}$$

are sub-matrices of A.

10.3. Equality of Matrices

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are equal, if and only if

(*i*) they are of the same order *i.e.* $m = p$ and $n = q$

(*ii*) their corresponding elements are **all** equal *i.e.*, $a_{ij} = b_{ij}$ for all *i* and *j*.

If A and B are two equal matrices, then we write A = B.

10.4. Addition (sum) of two Matrices

We can add two matrices only when they are of the **same order** and two such matrices are said to be **conformable for addition.**

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices of the same order $m \times n$, then **their sum A + B** *is a matrix of the same order m × n and is obtained by adding the corresponding elements of A and B.*

Thus, if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, then the sum

$$A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}.$$

Remarks. The elements of a matrix will be assumed to belong to some number system say of Rationals, Reals or Complex.

10.5. Properties of Matrix Addition

1. Matrix Addition is Commutative. *i.e.* if A and B are matrices of the same order, then **A + B = B + A.** *(M.D.U. 1983)*

Proof. L.H.S. = A + B

$$= [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [(a_{ij} + b_{ij})]_{m \times n}$$
$$= [(b_{ij} + a_{ij})]_{m \times n}$$

[∵ Elements of matrices are commutative for addition]

$$= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} = B + A = \text{R.H.S.}$$

2. Matrix Addition is Associative. If A, B, C be matrices of the same order, then (A + B) + C = A + (B + C).

Proof. L.H.S. = (A + B) + C

$$= ([a_{ij}]_{m \times n} + [b_{ij}]_{m \times n}) + [c_{ij}]_{m \times n}$$
$$= [(a_{ij} + b_{ij})]_{m \times n} + [c_{ij}]_{m \times n}$$
$$= [(a_{ij} + b_{ij}) + c_{ij}]_{m \times n}$$
$$= [a_{ij} + (b_{ij} + c_{ij})]_{m \times n}$$

[∵ For elements of matrices, addition is associative]

$$= [a_{ij}]_{m \times n} + ([b_{ij}]_{m \times n} + [c_{ij}]_{m \times n})$$
$$= A + (B + C) = \text{R.H.S.}$$

Note. Because of associative property of addition, we write

(A + B) + C = A + (B + C) = A + B + C.

3. Existence of Additive identity. Given any matrix A of order $m \times n$, there exists a matrix O of order $m \times n$, each of whose element is zero such that A + O = A.

Note. The zero matrix O is called additive identity or a zero and is unique for a set of all $m \times n$ matrices.

4. Existence of Additive Inverse. Given a matrix A of order $m \times n$; their exists a matrix X also of the same order, so that

A + X = O

This matrix $X = -[a_{ij}]$ **is called additive inverse or Negative of A** and we shall denote it by (– A).

Thus if $A = [a_{ij}]$, then $-A = [-a_{ij}]$.

Proofs of (3) and (4) are left to the reader as an exercise.

10.6. Subtraction of Two Matrices

Let A and B be two matrices of the same order (type), then subtraction of B from A is written as A – B and is defined as sum of A and – B.

Thus, as A – B = A + (– B).

Hence A – B is obtained by subtracting from each element of A the corresponding element of B.

10.7. Multiplication of a Matrix by a Scalar

Let $A = [a_{ij}]_{m \times n}$ be any matrix and k any scalar, then the multiplication of A by the scalar k written as k A is a matrix of order $m \times n$ obtained by multiplying each element of A by the scalar k. Thus,

If $\quad A = [a_{ij}]_{m \times n}$, then

$$kA = k[a_{ij}]_{m \times n} = [k.a_{ij}]_{m \times n}.$$

For example. If $A = \begin{bmatrix} -1 & 2 & 7 & 8 \\ 3 & 4 & -2 & 7 \\ 1 & 2 & 3 & 4i \end{bmatrix}$ is a matrix of order 3 × 4

and 5 is a scalar, then

$$5A = \begin{bmatrix} -5 & 10 & 35 & 40 \\ 15 & 20 & -10 & 35 \\ 5 & 10 & 15 & 20i \end{bmatrix}.$$

10.8. Properties of Multiplication of a Matrix by a Scalar

If $A = [a_{ij}]$ and $B = [b_{ij}]$ be any two matrices of the same type $m \times n$ and x and y are scalars, then

(*i*) $x(A + B) = xA + xB$

(*ii*) $(x + y) A = xA + yA$

(*iii*) $x(yA) = (xy)A$

(*iv*) There exist a scalar 1 so that 1.A = A.

Proofs are easy and are left as an exercise to the readers.

10.9 Multiplication of Two Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices, then the produced AB in this order is defined if the number of columns in A (pre-factor) = the number of rows in B (post-factor), and

(*i*) Number of rows in AB = the number of rows in A.

(*ii*) Number of columns in AB = the number of cols. in B.

(*iii*) The (i, j)th element of AB = sum of products of the elements of ith row of A with the corresponding elements of the jth column of B.

In Symbols. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices, then the product AB is defined and is a matrix of order $m \times p$.

Let $\quad AB = C = [c_{ij}]_{m \times p}$, where

$$c_{ij} = (i, j)\text{th element of } C(= AB)$$

$$= (i\text{th row of A}) \begin{pmatrix} j\text{th} \\ \text{col.} \\ \text{of} \\ \text{B} \end{pmatrix}$$

$$= (a_{i1}\ a_{i2}\ \ldots\ldots\ a_{in}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \ldots \\ b_{nj} \end{pmatrix}$$

$$= a_{i1}b_{1j} + a_{2i}b_{2j} + \ldots\ldots + a_{in}b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Remarks. 1. If the product AB is defined, then the matrices A and B are said to be conformable for multiplication AB.

2. If AB is defined, BA may or may not be defined.

3. Method of multiplication is known as Row-by-Column method.

10.10. Properties of Matrix Multiplication

Property 1. Matrix Multiplication is associative. *If A, B, C are matrices of the order $m \times n$, $n \times p$, $p \times q$ respectively, then*

$$(AB)C = A(BC).$$

Proof. A is a matrix of order $m \times n$, B is of order $n \times p$.

∴ AB is a matrix of order $m \times p$; C is a matrix of order $p \times q$.

∴ (AB)C is a matrix of type $m \times q$.

Similarly, it is easy to see that A(BC) is a matrix of order $m \times q$.

Thus (AB)C and A(BC) are matrices both of the same order. ...(1)

Let $\quad A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$, $C = [c_{ij}]_{p \times q}$.

Now (i, k)th element of the product AB

= sum of the products of elements of ith row of A and kth col. of B

$$= \sum_{l=1}^{n} a_{il}b_{lk} = d_{ik} \text{ (say)}$$

Now (i, j)th element of the product (AB)C

= sum of the products of elements of ith row of AB and jth column of C

$$= \sum_{k=1}^{p} d_{ik}c_{kj}$$

$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} a_{il}b_{lk} \right) c_{kj}$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} (a_{il} b_{lk}) c_{kj}$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} a_{il} (b_{lk} c_{kj})$$

[∵ Multiplication is associative for elements of matrices]

$$= \sum_{l=1}^{n} a_{il} \sum_{k=1}^{p} (b_{lk} c_{kj})$$

= Sum of the products of elements of ith row of A with jth column of BC

= (i, j)th element of A(BC) ...(2)

From (1) and (2),

(AB)C = A(BC).

Note. (AB)C and A(BC) both are written = ABC.

Property 2. Distributive Laws :

If A, B, C are three matrices of type $m \times n$, $n \times p$, $n \times p$ respectively, then

(*i*) A(B + C) = AB + AC [*Left* : Distributive Law] (*M.D.U 1995*)

(*ii*) (B + C)A = BA + CA [*Right* : Distributive Law]

To prove A(B + C) = AB + AC.

A is a matrix of order $m \times n$ and (B + C) is a matrix of order $n \times p$, therefore A(B + C) is a matrix of order $m \times p$. Similarly, each of the matrix AB, AC is of order $m \times p$.

∴ AB + AC is a matrix of order $m \times p$.

∴ A(B + C) and AB + AC are matrices of the same order. ...(1)

Now (i, j)th element of A(B + C)

$$= \sum_{k=1}^{n} a_{ik} (b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj})$$

[Using distributive law for elements]

$$= \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

= (i, j)th ele. of AB + (i, j)th ele. of AC

= (i, j)th ele. of (AB + AC) ...(2)

From (1) and (2),

A(B + C) = AB + AC.

Similarly (A + B)(C) = AC + BC.

Property 3. If A be any $n \times n$ matrix, then

$AI_n = A = I_nA$. Proof is left to the reader as an exercise.

Property 4. Matrix Multiplication is not commutative.

Prove that the product of matrices is not commutative in general i.e., prove AB ≠ BA, discussing all possibilities.

Proof. Case I. AB is defined but BA is not defined.

Let A be of order 3×2 and B be of order 2×4.

∴ AB is defined and is a matrix of order 3×4.

But BA is not defined ∴ AB ≠ BA.

Case. II. AB and BA are both defined but are of different order.

Let A be of order 2×3 and matrix B of order 3×2.

∴ AB is defined and is a matrix of order 2×2.

BA is also defined and is a matrix of order 3×3.

∴ AB ≠ BA.

Case III. *AB and BA are both defined and both are of the same order, yet AB ≠ BA.*

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix}$

be two square matrices of the same order 2×2.

$$\therefore \quad AB = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 4+21 & 2+15 \\ -2+28 & -1+20 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 17 \\ 26 & 19 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 4-1 & 6+4 \\ 14-5 & 21+20 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 10 \\ 9 & 41 \end{bmatrix}$$

Thus, AB ≠ BA.

Property 5. Give an example of matrices A and B such that A ≠ 0, B ≠ 0, but AB = 0.

Or **Prove that AB = 0, does not imply either A = 0 or B = 0.**

Proof. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

∴ A ≠ 0, B ≠ 0

$$AB = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 0+0 \\ 1-1 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $A \neq 0$, $B \neq 0$ yet $AB = 0$.

Note. If $AB = 0$, BA may or may not be zero.

Property 6. Cancellation law in multiplication does not hold i.e. Give an example of matrices A, B, C such that

$AB = AC$ but $B \neq C$

Proof. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3+2 & 1+1 \\ 3+2 & 1+1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2+3 & 1+1 \\ 2+3 & 1+1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}$$

Thus, $AB = AC$, whereas $B \neq C$.

10.11. Positive Integral Powers of a Matrix

If A is a square matrix, then A . A is also a square matrix of the same order and we write $A.A = A^2$; $A.A.A = A^3$ etc.

For all positive integers m and n, the following results hold

(1) $A^m = A \cdot A \cdot A \ldots$ m times

(2) $A^m \cdot A^n = A^{m+n}$

(3) $(A^m)^n = A^{mn}$

(4) **We define that $A^0 = I$.**

Example 1. *If* $A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$. *Show that* $A^2 = 0$.

Sol.
$$A^2 = A \cdot A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0 + 0.2 & 0.0 + 0.0 \\ 2.0 + 0.2 & 2.0 + 0.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $A^2 = 0$, whereas $A \neq 0$.

Note. Hence from $A^2 = 0$, we cannot conclude that $A = 0$.

Example 2. *If* $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}$, *find AB and BA and show that* $AB \neq BA$.

Sol. Here A is 3×3 matrix and B is also 3×3 matrix.

$\therefore$ AB and BA are both defined and are matrices of the same order 3×3.

$$AB = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 + 3.1 + 0.0 & 1.1 + 3.0 + 0.5 & 1.0 + 3.0 + 0.1 \\ 1.0 + 1.1 + 0.0 & 1.1 + 1.0 + 0.5 & 1.0 + 1.0 + 0.1 \\ 4.0 + 1.1 + 0.0 & 4.1 + 1.0 + 0.5 & 4.0 + 1.0 + 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 + 1.1 + 0.4 & 0.3 + 1.1 + 0.1 & 0.0 + 1.0 + 0.0 \\ 1.1 + 0.1 + 0.4 & 1.3 + 0.1 + 0.1 & 1.0 + 0.0 + 0.0 \\ 0.1 + 5.1 + 1.4 & 0.3 + 5.1 + 1.1 & 0.0 + 5.0 + 1.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 9 & 6 & 0 \end{bmatrix}$$ **Hence $AB \neq BA$.**

Example 3. *By using Principle of Mathematical Induction prove that if*

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}, \text{ then}$$

$$A^n = \begin{bmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{bmatrix}$$

n being positive integer. (K.U. 1991 S)

Sol. The result to be proved is

$$A^n = \begin{bmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{bmatrix} \quad \ldots(1)$$

Putting $n = 1$ in (1), we get

$$A = \begin{bmatrix} 1 + 2 & -4 \\ 1 & 1 - 2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

which shows that result is proved for $n = 1$.

Let us assume the result to be true for $n = k$

i.e.,
$$A^k = \begin{bmatrix} 1 + 2k & -4k \\ k & 1 - 2k \end{bmatrix}$$

we shall, prove the result for $n = k + 1$ i.e.

$$A^{k+1} = \begin{bmatrix} 1 + 2(k+1) & -4(k+1) \\ k + 1 & 1 - 2(k+1) \end{bmatrix} \quad \ldots(2)$$

L.H.S. of (2) $= A^{k+1} = A^k \cdot A$

$$= \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2k)3 + (-4k)(1) & (1+2k)(-4) + (-4k)(-1) \\ k.3 + 1(1-2k)1 & k(-4) + (1-2k)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix} = \text{R.H.S. of (2).}$$

Thus, the result is true for $n = k + 1$, whenever it is true for $n = k$.

Hence by induction the result is true for all positive integers n.

Example 4. *Define the following and give one example of each :*

(*i*) *Idempotent matrix*

(*ii*) *Nilpotent matrix*

(*iii*) *Involutory matrix.*

Sol. (*i*) *A square matrix A is said to be Idempotent if $A^2 = A$.*

For example, the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

(Verify that $A^2 = A$).

(*ii*) *A square matrix A is called Nilpotent if there exists a positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A.*

For example, the matrices $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ are nilpotent

(Verify that $A^2 = O$).

Every upper triangular matrix is nilpotent.

(*iii*) *A square matrix A is said to be Involutory if $A^2 = I$.*

For example, the matrix $A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & -\sqrt{2} \end{bmatrix}$ is involutory.

Example 5. *Show that the matrix A is involutory, if and only if*

$$(I + A)(I - A) = O.$$

Sol. Let A be an involutory matrix of order n.

Then $\quad A^2 = I$

$\Rightarrow \quad A^2 - I = O$

$\Rightarrow \quad I^2 - A^2 = O \qquad (\because I^2 = I)$

$\Rightarrow \quad (I - A)(I + A) = O \qquad (\because AI = IA)$

Conversely, if $(I + A)(I - A) = O$

then $\quad I^2 - IA + AI - A^2 = O$

$\Rightarrow \quad I - A^2 + AI - AI = O \qquad (\because IA = AI)$

$\Rightarrow \quad I - A^2 + O = O$

$\Rightarrow \quad I - A^2 = O$

$\Rightarrow \quad A^2 = I.$

EXERCISE 10 (*a*)

1. Perform matrix multiplication AB, where

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

2. If $\quad A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$,

then show that

$$A_\alpha . A_\beta = A_{\alpha+\beta} = A_\beta . A_\alpha.$$

3. Find the product of the matrices :

$$\begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}, \text{ where } i^2 = -1.$$

4. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Find AB, BA. Is AB = BA ? (*M.D.U. 1981 S*)

5. Show that for

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$A^2 = O.$$

6. If $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, where $i^2 = -1$.

Verify that $\quad (A + B)^2 = A^2 + B^2$.

7. (*a*) Show that the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

is a solution of the matrix equation

$$A^2 - 5A + 7I = O. \qquad (K.U.\ 1980\ ;\ M.D.U.\ 1983)$$

(*b*) If $f(x) = x^2 - 5x + 7$, find $f(A)$, where

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}. \qquad (K.U.\ 1988)$$

8. If $f(x) = x^2 - 5x + 6$, find $f(A)$, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}. \qquad (K.U.\ 1980)$$

[**Hint.** $f(A) = A^2 - 5A + 6I_3$, find A^2 and substitute the values of I_3, A and A^2.]

9. Prove that matrix multiplication is distributive over matrix addition. *(M.D.U. 1991)*

(Reproduce property 2, Art. 10.10)

10. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Show that

$$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}.$$ *(M.D.U. 1994 ; G.N.D.U. 1981)*

11. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, prove that

$$(aI_2 + bA)^n = a^n I_2 + na^{n-1} bA$$

for a positive integer n. *(M.D.U. 1993)*

12. Determine all the idempotent diagonal matrices of order n.

[**Hint.** If $A^2 = \begin{bmatrix} d_1 & 0 & 0 \ldots\ldots 0 \\ 0 & d_2 & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots\ldots d_n \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \ldots\ldots 0 \\ 0 & d_2 & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots\ldots d_n \end{bmatrix}$

$$= \begin{bmatrix} d_1^2 & 0 & 0 \ldots\ldots 0 \\ 0 & d_2^2 & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots\ldots d_n^2 \end{bmatrix}$$

$\therefore \quad A^2 = A \Rightarrow d_i^2 = d_i$ for $i = 1, 2, \ldots\ldots, n$

$\Rightarrow \quad d_i = 0, 1$ for $i = 1, 2, \ldots\ldots, n$

Hence A = dia. $[d_1, d_2, \ldots\ldots, d_n]$ for $d_1, d_2, \ldots\ldots, d_n \in \{0, 1\}$ is the required idempotent diagonal matrix.]

13. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent with index 3.

[**Hint.** Show $A^3 = O$.]

14. Show that the following matrices are involutory

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

15. Show that the sum of two Idempotent matrices A and B is idempotent if AB = BA = O.

Answers

1. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

4. $AB = \begin{bmatrix} -1 & 12 & 11 \\ -1 & 7 & 8 \\ -2 & -1 & 5 \end{bmatrix}, BA = \begin{bmatrix} 5 & 9 & 13 \\ -1 & 2 & 4 \\ -2 & 2 & 4 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$.

9. $A = \begin{bmatrix} 2 & -3 & 4 \\ 5 & -7 & 8 \\ 3 & 4 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 10 \\ 9 \\ 15 \end{bmatrix}$.

10.12. Transpose of a matrix

Let A be any given matrix of the order $m \times n$, then a matrix obtained from A by changing its rows into columns and columns into rows is called the **transpose of a matrix A** and is denoted by A′ which will be of the type $n \times m$.

In symbols. If $A = [a_{ij}]_{m \times n}$, then

$$A' = [c_{ij}]_{n \times m}, \text{ where } c_{ij} = a_{ji}$$

i.e., (i, j)th element of A′ = (j, i)th element of A.

For example,

Let $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -3 & 4 & -5 \\ 6 & 7 & -8 & 2 \end{bmatrix}$

then $A' = \begin{bmatrix} 1 & 2 & 6 \\ 2 & -3 & 7 \\ 3 & 4 & -8 \\ -1 & -5 & 2 \end{bmatrix}$.

10.13. Theorem

If A′ and B′ denote transpose of A and B, prove that

(1) (A′)′ = A

(2) (A + B)′ = A′ + B′, A, B are conformable for addition

(3) (kA)′ = kA′, k is any scalar

(4) (AB)′ = B′A′, A, B are conformable for multiplication *(M.D.U. 1982)*

(5) $(A^n)' = (A')^n$, A is a square matrix, n is a positive integer.

Proof. (1) Let $A = [a_{ij}]_{m \times n}$

$\therefore \quad A' = [c_{ij}]_{n \times m}$, where $c_{ij} = a_{ji}$

$\therefore \quad (A')' = [d_{ij}]_{m \times n}$

where $d_{ij} = c_{ji} = a_{ij}$

$\therefore \quad (A')' = [a_{ij}]_{m \times n} = A.$

(2) Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$

$\therefore$ A + B is a matrix of order $m \times n$

$\therefore$ $(A + B)'$ is a matrix of order $n \times m$

Again A′ and B′ are matrices of order $n \times m$

$\therefore$ $(A' + B')$ is a matrix of order $n \times m$

$\therefore$ $(A + B)'$ and $(A' + B')$ are matrices of the same order.

(i, j)th element of $(A + B)'$

= (j, i)th element of (A + B)

= (j, i)th element A + (j, i)th element of B

= (i, j)th element of A′ + (i, j)th element of B′.

= (i, j)th element of (A′ + B′).

Thus $(A + B)' = A' + B'$.

(3) Let $A = [a_{ij}]_{m \times n}$

$\therefore$ $(kA)'$ and kA' are matrices of the same order $n \times m$. (i, j)th element of $(kA)'$

= (j, i)th element of kA

= $k\,[(j, i)$th element of A$]$

= $k\,[(i, j)$th element of A′$]$

= (i, j)th element of kA'

$\therefore$ $(kA)' = kA'$.

(4) *To prove* $(AB)' = B'A'$

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$

$\therefore$ $A' = [\alpha_{ij}]_{n \times m}$ and $B' = [\beta_{ij}]_{p \times n}$, where

$\alpha_{ij} = a_{ji}$ and $\beta_{ij} = b_{ji}$

Now AB is a matrix of order $m \times p$.

$\therefore$ $(AB)'$ is a matrix of order $p \times m$. Also $B'A'$ is a matrix of order $p \times m$. ...(1)

$\therefore$ $(AB)'$ and $B'A'$ are matrices of the same order $p \times m$.

(i, j)th element of $(AB)'$

= (j, i)th element of AB

$$= \sum_{k=1}^{n} a_{jk} b_{ki}$$ (The sum of products of elements of jth row of A with corresp. element of ith col. of B)

$$= \sum_{k=1}^{n} \alpha_{kj} \beta_{ik} \qquad [\because \alpha_{ij} = a_{ji} \text{ and } \beta_{ij} = b_{ji}]$$

$$= \sum_{k=1}^{n} \beta_{ik} \alpha_{kj}$$

= (i, j)th element of $B'A'$ ...(2)

$\therefore$ From (1) and (2),

$$(AB)' = B'A'$$

Cor. $(A_1 . A_2 \dots A_n)' = A_n' . A_{n-1}' \dots A_2' . A_1'$

Putting $A_1 = A_2 \dots = A_n = A$, where A is a sq. matrix

$\therefore$ $(A . A \dots A)' = A' . A' \dots A' . A'$

$\therefore$ $(A^n)' = (A')^n$

Hence $(A^n)' = (A')^n$, n being a natural number.

10.14. Conjugate of a matrix

Let A be a given matrix of order $m \times n$ over the complex number system, then a matrix obtained from A by replacing each of its elements by their corresponding complex conjugates is called the **conjugate of A** and is denoted by $\overline{A}$, where $\overline{A}$ is also of the same order $m \times n$.

In notation we can define as

If $A = [a_{ij}]_{m \times n}$, then

$\overline{A} = [b_{ij}]_{m \times n}$, where $b_{ij} = \overline{a_{ij}}$

For example,

Let $$A = \begin{bmatrix} 2+i & 2 & 5i \\ 5i+7 & -8 & 4i-3 \\ 2 & 5+i & 4-2i \end{bmatrix}$$

$\therefore$ $$\overline{A} = \begin{bmatrix} 2-i & 2 & -5i \\ -5i+7 & -8 & -4i-3 \\ 2 & 5-i & 4+2i \end{bmatrix}$$

It is to be noted that conjugate complex of $5i + 7$ is $-5i + 7$.

10.15. Theorem

If $\overline{A}$ and $\overline{B}$ denote the conjugate of A and B, respectively, then prove that

1. $\overline{(\overline{A})} = A$.
2. $\overline{(A + B)} = \overline{A} + \overline{B}$, where A and B are conformable for addition.
3. $\overline{(kA)} = \overline{k}\,\overline{A}$, where k is any complex number.
4. $\overline{(AB)} = \overline{A} . \overline{B}$.
5. $(\overline{A})^n = \overline{(A^n)}$.

Proofs. Proofs for properties (1), (2) and (3) are easy and are left to the reader as an exercise.

4. *To prove* $\overline{AB} = \overline{A} . \overline{B}$.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, where the elements a_{ij} and b_{ij} are over the complex field.

$\therefore \quad \overline{A} = [\alpha_{ij}]_{m \times n}$ where $\alpha_{ij} = \overline{a_{ij}}$

and $\quad \overline{B} = [\beta_{ij}]_{n \times p}$ where $\alpha_{ij} = \overline{b_{ij}}$.

Now $\overline{AB}$ and $\overline{A} \,.\, \overline{B}$ are matrices, both of the same order $m \times p$.

(i, j)th element of $\overline{AB}$ ...(1)

$$= \text{Conjugate of the } (i, j)\text{th element of AB}$$

$$= \overline{\left(\sum_{k=1}^{n} a_{ik} \,.\, b_{kj} \right)}$$

$$= \sum_{k=1}^{n} \overline{a_{ik} \,.\, b_{kj}} \qquad [\text{Using } \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}]$$

$$= \sum_{k=1}^{n} \overline{a_{ik}} \,.\, \overline{b_{kj}} \qquad [\text{Using } \overline{z_1 z_2} = \overline{z_1} \,.\, \overline{z_2}]$$

$$= \sum_{k=1}^{n} \alpha_{ik} \,.\, \beta_{kj}$$

$$= (i, j)\text{th element of } \overline{A} \,.\, \overline{B} \qquad ...(2)$$

From (1) and (2),

$$\overline{\mathbf{AB}} = \overline{\mathbf{A}} \,.\, \overline{\mathbf{B}}.$$

(5) To prove $\overline{A^n} = (\overline{A})^n$

Using the above result (4),

$$\overline{A_1 \,.\, A_2 \,......\, A_n} = \overline{A_1} \,.\, \overline{A_2} \,......\, \overline{A_n}$$

where the product on each side is defined.

Put $\quad A_1 = A_2 = A_n = A$

$$\therefore \quad \overline{A \,.\, A \,.\, A \,......\, n \text{ terms}} = \overline{A} \,.\, \overline{A} \,.\, \overline{A} \,......\, n \text{ terms}$$

$\therefore \quad \overline{A^n} = (\overline{A})^n$, n is a natural number.

10.16. Transposed Conjugate of a Matrix

The transposed of the conjugate or conjugate of the transpose of a matrix A is called *Transposed Conjugate* of A and is denoted by A^θ or by A^*. Thus

$$A^\theta = (\overline{A})' = (\overline{A'}).$$

Thus if $A = [a_{ij}]$, then $A^\theta = [\alpha_{ij}]$ where $\alpha_{ij} = \overline{a_{ji}}$

i.e. (i, j)th element of A^θ = The conjugate complex of the (j, i)th element of A. For example, if

$$A = \begin{bmatrix} 1 - 2i & 2 + 3i & 4 \\ -7 & 8i & 5 \\ 0 & 6i + 5 & 4 \end{bmatrix}$$

then
$$\overline{A} = \begin{bmatrix} 1 + 2i & 2 - 3i & 4 \\ -7 & -8i & 5 \\ 0 & -6i + 5 & 4 \end{bmatrix}$$

$$A^\theta = (\overline{A})' = \begin{bmatrix} 1 + 2i & -7 & 0 \\ 2 - 3i & -8i & -6i + 5 \\ 4 & 5 & 4 \end{bmatrix}.$$

10.17. Theorem. If A^θ and B^θ be the transposed conjugate of A and B respectively, then

1. $(A^\theta)^\theta = A$

2. $(A + B)^\theta = A^\theta + B^\theta$, A, B are of the same order

3. $(kA)^\theta = \overline{k}A^\theta$, k is any complex number

4. $(AB)^\theta = B^\theta A^\theta$, A, B are conformable for multiplication.

Proof.

1. $\quad (A^\theta)^\theta = \overline{((\overline{A})')'} = (\overline{\overline{A}}) = A$

2. $\quad (A + B)^\theta = ((\overline{A + B}))' = (\overline{A} + \overline{B})'$

$$= (\overline{A})' + (\overline{B})' = A^\theta + B^\theta$$

3. $\quad (kA)^\theta = (\overline{kA})' = (\overline{k}\,\overline{A})' = \overline{k}\,(\overline{A})'$

$$= \overline{k}\,A^\theta$$

4. $\quad (AB)^\theta = (\overline{AB})' = (\overline{A}\,\overline{B})'$

$$= (\overline{B})'\,(\overline{A})' = B^\theta A^\theta.$$

10.18. Symmetric Matrix

Def. A matrix A is said to be a symmetric matrix if $A' = A$, *i.e.* if the transpose of a matrix is equal to itself.

Let $\quad A = [a_{ij}]_{m \times n}$

$\therefore \quad A' = [\alpha_{ij}]_{n \times m}$, where $\alpha_{ij} = a_{ji}$.

The matrix A will be symmetric, if and only if,

$$A = A'$$

i.e. if and only if $m = n$ and $a_{ij} = \alpha_{ij} = a_{ji}$. Thus we have

Definition. *A square matrix $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j, i.e.*

A square matrix is symmetric if and only if (i, j)th element = (j, i)th element.

EXERCISE 10 (*b*)

1. Define the following and give one suitable example in each case :
   (*i*) Transpose of a matrix (*ii*) Symmetric matrix
   (*iii*) Skew-symmetric matrix (*iv*) Hermitian matrix
   (*v*) Skew-Hermitian matrix.
2. Find the transpose of the following matrices and point out if any of them is symmetric or Skew-symmetric

   (*i*) $\begin{bmatrix} a & b & c \\ b & k & m \\ c & m & x \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 11 \\ -7 & -11 & 0 \end{bmatrix}$.

3. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $B = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $i^2 = -1$

   Verify that $(AB)' = B'A'$.

4. If $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 7 \\ 0 & -5 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 6 \\ 0 & -2 & -4 \\ -6 & 8 & -8 \end{bmatrix}$,

   Verify $(AB)' = B'A'$. (*M.D.U. 1994*)

5. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -7 \end{bmatrix}$

   Verify that $(A^2)' = (A')^2$.

6. If $A = \begin{bmatrix} 2+3i & i \\ 6i+5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} i & 2i+1 \\ 2-i & -i \end{bmatrix}$

   Verify that $\overline{(AB)} = \overline{A} \, . \, \overline{B}$.

7. Prove by an example of a matrix 3 × 3, that if A is a lower triangular matrix, then A′ is an upper triangle matrix.
8. If A and B are symmetric, show that (AB + BA) is symmetric and (AB – BA) is skew-symmetric.
9. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $B = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$,

   verify that $(AB)^{\theta} = B^{\theta} A^{\theta}$. (*M.D.U. 1993*)

10. Show that the matrix $\begin{bmatrix} 0 & 1+i & 2+3i \\ 1-i & 1 & -i \\ 2-3i & -2 & 0 \end{bmatrix}$ is Hermitian. (*M.D.U. 1993*)

11. Show that (*i*) $A = \begin{bmatrix} 2 & 1+i & 2+3i \\ 1-i & 1 & -i \\ 2-3i & i & 0 \end{bmatrix}$ is Hermitian.

    (*ii*) $B = \begin{bmatrix} 2i & 1+i & 2-3i \\ -1+i & 5i & 2 \\ -2-3i & -2 & 0 \end{bmatrix}$ is Skew-Hermitian. (*M.D.U. 1994*)

    (*iii*) *i*B is Hermitian.

12. If A is a Hermitian (Skew-Hermitian) matrix, then show that ± *i*A is Skew-Hermitian (Hermitian).
13. Show that every square matrix can be uniquely expressed as the sum of a Hermitian and a Skew-Hermitian matrix. (*K.U. 1988*)

    [**Hint.** Write $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta}) = P + Q$ and show that $P^{\theta} = P$ and $Q^{\theta} = -Q$.]

14. If A and B are Hermitian, show that
    (*i*) AB + BA is Hermitian (*K.U. 1991 S*)
    (*ii*) AB – BA is Skew-Hermitian. (*K.U. 1991 S*)
    (*iii*) AB is Hermitian if and only if AB = BA.
    (*iv*) BAB and ABA are Hermitian.

Answers

2. (*i*) $\begin{bmatrix} a & b & c \\ b & k & m \\ c & m & x \end{bmatrix}$ (It is a Symmetric Matrix)

   (*ii*) $\begin{bmatrix} 0 & -5 & -7 \\ 5 & 0 & -11 \\ 7 & 11 & 0 \end{bmatrix}$ (It is a Skew-symmetric Matrix).

10.20. Definition. Determinant of a Square Matrix

(*i*) If $A = [a_{11}]$ is a square matrix of order 1 × 1 over a field F, then determinant of the matrix A is the number $a_{11} \in F$. Thus

$$\det A = |A| = a_{11}$$

(*ii*) If $A = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots \\ a_{n1} & a_{n2} \cdots\cdots a_{nn} \end{bmatrix}$

be a square matrix of order $n \times n$ over a field F, where $n \geq 2$, then we write determinant of A as

$$\det A = |A|$$

$$= \begin{vmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ a_{n1} & a_{n2} \cdots\cdots a_{nn} \end{vmatrix}$$

$$= (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \ldots\ldots + (-1)^{i+n} a_{in} \det A_{in}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det A_{ij} \quad \ldots(1)$$

where A_{ij} is a sub-matrix of order $(n-1) \times (n-1)$, obtained by deleting the ith row and the jth col. of matrix A and determinant of A_{ij} is defined by applying induction on n.

$$\det A = \sum_{j=1}^{n} a_{ij} \cdot (-1)^{i+j} \det A_{ij}$$ is called expansion of det A by the ith row. Similarly, we can write

$$\det A = \sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+j} \det A_{ij}.$$

It is known as expansion of det A by the jth column.

We observe that det A is a scalar $\in F$. Thus, a determinant is a function on the set of all $n \times n$ square matrices over the field F.

10.21. Definition. Minor of an Element

If $A = [a_{ij}]$ is any square matrix, then det A_{ij} called the *minor* of (i, j)th entry a_{ij} of A and may be denoted by M_{ij}.

10.22. Co-factor of an Element

If $A = [a_{ij}]$ is any square matrix of order $n \times n$, then $(-1)^{i+j} \det A_{ij}$ is called the co-factor of (i, j)th entry A_{ij} of A, and may be denoted by C_{ij}. Thus

C_{ij} = Co-factor of (i, j)th entry of a matrix A

$= (-1)^{i+j} \det A_{ij}$, where A_{ij} is the $(n-1) \times (n-1)$

sub-matrix of A, obtained by deleting the ith row and the jth col. of A.

Remarks :

In terms of co-factors, the expansion of the determinant of a square matrix $A = [a_{ij}]_{n \times n}$ is

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \text{ (expansion by } i\text{th row)}$$

$$= \sum_{i=1}^{n} a_{ij} C_{ij} \text{ (expansion by } j\text{th col.)}$$

10.23. An important Property

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

(*i*) $\sum_{j=1}^{3} a_{ij} C_{rj} = \det A$ if $r = i$.

(*ii*) $\sum_{j=1}^{3} a_{ij} C_{rj} = 0$ if $r \neq i$.

Proof. (*i*) When $r = i$,

$$\sum_{j=1}^{3} a_{ij} C_{ij}$$

$= a_{i1} C_{i1} + a_{i2} C_{i2} + a_{i3} C_{i3}$ [Take $i = 1$ (or 2 or 3)]

$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= a_{11} \cdot (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \cdot (-1)^3 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$+ a_{13} \cdot (-1)^4 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

$= \det A.$

(*ii*) $r \neq i$

$$\sum_{j=1}^{3} a_{ij} C_{rj} = a_{i1}C_{r1} + a_{i2}C_{r2} + a_{i3}C_{r3}$$

Taking $i = 1$ and $r = 2$ (say)

$= a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$

$= a_{11} \cdot (-1)^3 \det A_{21} + a_{12} \cdot (-1)^4 \det A_{22}$

$+ a_{13} \cdot (-1)^5 \det A_{23}$

$= -a_{11} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{13} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

$= -a_{11}(a_{12} \cdot a_{33} - a_{13} \cdot a_{32}) + a_{12}(a_{11} \cdot a_{33} - a_{31}a_{13})$

$- a_{13}(a_{11} \cdot a_{32} - a_{12} \cdot a_{31})$

$= -a_{11}a_{12}a_{33} + a_{11}a_{13}a_{32} + a_{11}a_{12}a_{33} - a_{12}a_{31}a_{13}$

$- a_{11}a_{13}a_{32} + a_{13}a_{12}a_{31}$

$= 0.$

Remark. *The result is quite general and holds for determinants of square matrices of all order. Thus if* $A = [a_{ij}]_{n \times n}$, *then*

(1) $\sum_{j=1}^{n} a_{ij}C_{rj} = \det A$ if $i = r$

$= 0$ if $i \neq r$.

(2) *If A is a square matrix with any one line consisting of zero elements, then*

det A = 0 [$\because$ $a_{ij} = 0$ for some i or j]

(3) *If A is triangular matrix, then*

det A = product of the diagonal elements.

$$= \begin{vmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_1 & 0 \\ 0 & 0 & \Delta_1 \end{vmatrix} \qquad [\because a_1A_1 + b_1B_1 + c_1C_1 = \Delta$$
$$a_1A_2 + b_1B_2 + c_1C_2 = 0 \text{ etc.}]$$

$$= {\Delta_1}^3$$

$\therefore \quad \Delta_2 = {\Delta_1}^2$

Hence $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$.

Adjugate determinants *or Reciprocal determinants*. If all the elements in a determinant Δ be replaced by their co-factors in Δ, then the determinant so obtained is called **Adjugate or Reciprocal of Δ.**

For example in the above example Δ_2 is adjugate of Δ_1. In general if Δ_1 is of nth order, then $\Delta_2 = {\Delta_1}^{n-1}$.

EXERCISE 10 (*d*)

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \\ 0 & 7 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 6 & 7 \\ 5 & 1 & 1 \end{bmatrix}$

   Verify that det. (AB) = (det. A) (det. B).

2. If $A = \begin{bmatrix} 1+i & i & 5i+2 \\ 5 & -1 & 0 \\ 2+i & 1 & 7 \end{bmatrix}$

   Verify that det. $\overline{A} = \overline{(\text{det. A})}$

   where the bar indicates the complex conjugate.

3. If $AA' = I$, then $|A| = \pm 1$.

4. I Δ' is the reciprocal determinant of a determinant Δ of order n, then $\Delta' = \Delta^{n-1}$, (proceed as in example 5).

5. Prove that if A and B are two square matrices of order n, then

   (*i*) $|A'B| = |AB'| = |A'B'| = |AB|$

   (*ii*) $|A^\theta B^\theta| = |\overline{AB}|$.

10.27. Adjoint of a Square Matrix

Def. *If $A = [a_{ij}]$ is a square matrix of order n, and A_{ij} is the cofactor of a_{ij} in $|A|$, then the matrix*

$$[A_{ij}]' = \begin{bmatrix} A_{11} & A_{21} \ldots\ldots & A_{n1} \\ A_{12} & A_{22} \ldots\ldots & A_{n2} \\ \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} \ldots\ldots & A_{nn} \end{bmatrix}$$

is called the adjoint of A and is written as adj. A.

In other words. To find adjoint of square matrix A, replace each element of A by its co-factor in | A | and take the transpose, the matrix so obtained will be adjoint of A.

Example 1. *Calculate the adjoint of A*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Sol. $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\therefore$ Co-factor of a, (1, 1)th element $= (-1)^2 |d| = d$

Co-factor of b, (1, 2)th element $= (-1)^3 |c| = -c$

Co-factor of c, (2, 1)th element $= (-1)^3 |b| = -b$

Co-factor of d, (2, 2)th element $= (-1)^4 |a| = a$

$\therefore$ Adj. of $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}' = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 2. *Calculate the adjoint of the diagonal matrix*

$$A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}.$$

Sol. Co-factor of $d_1 = (-1)^2 \begin{vmatrix} d_2 & 0 \\ 0 & d_3 \end{vmatrix} = d_2d_3$

Co-factor of $d_2 = (-)^4 \begin{vmatrix} d_1 & 0 \\ 0 & d_3 \end{vmatrix} = d_1d_3$

Co-factor of $d_3 = (-1)^6 \begin{vmatrix} d_1 & 0 \\ 0 & d_2 \end{vmatrix} = d_1d_2$

$\therefore$ Adj. of $A = \begin{bmatrix} d_2d_3 & 0 & 0 \\ 0 & d_3d_1 & 0 \\ 0 & 0 & d_1d_2 \end{bmatrix}' = \begin{bmatrix} d_2d_3 & 0 & 0 \\ 0 & d_3d_1 & 0 \\ 0 & 0 & d_1d_2 \end{bmatrix}$

This shows that adjoint of a diagonal matrix is a diagonal matrix.

Example 3. *Calculate the adjoint of A, where*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{bmatrix}.$$

Sol. Co-factor of 1, (1, 1)th element

$$= (-1)^2 \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} = (-1-2) = -3$$

Co-factor of 2, (1, 2)th element

$$= (-1)^3 \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = -(1+4) = -5$$

Co-factor of 3, (1, 3)th element

$$= (-1)^4 \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = (1-2) = -1$$

Co-factor of 1, (2, 1)th element

$$= (-1)^3 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -(2-3) = 1$$

Co-factor of – 1, (2, 2)th element

$$= (-1)^4 \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} = (1+6) = 7$$

Co-factor of 2, (2, 3)th element

$$= (-1)^5 \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = -(1+4) = -5$$

Co-factor of – 2, (3, 1)th element

$$= (-1)^4 \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} = (4+3) = 7$$

Co-factor of 1, (3, 2)th element

$$= (-1)^5 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -(2-3) = 1$$

Co-factor of 1, (3, 3)th element

$$= (-1)^6 \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = (-1-2) = -3$$

$$\therefore \text{ Adj. of A} = \begin{bmatrix} -3 & -5 & -1 \\ 1 & 7 & -5 \\ 7 & 1 & -3 \end{bmatrix}' = \begin{bmatrix} -3 & 1 & 7 \\ -5 & 7 & 1 \\ -1 & -5 & -3 \end{bmatrix}.$$

10.28. Theorem. If A be a n-square matrix, then prove that

A(adj. A) = (adj. A) A = | A | I_n *(K.U. 1995 S)*

where I_n denotes the unit matrix of order n.

Proof. Let A = $[a_{ij}]$ be n-square matrix.

∴ Adj A = $[A_{ij}]'$, where A_{ij} is co-factor of a_{ij} in | A |

= $[\alpha_{ij}]$, where $\alpha_{ij} = A_{ji}$

To find the product of A (adi. A)

(i, j)th element of A (adj. A)

= sum of the products of the corresponding elements of the ith row of A and the jth col. of adj. A

$$= [a_{i1} a_{i2} \ldots . a_{in}] \begin{Bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \alpha_{3j} \\ \vdots \\ \alpha_{nj} \end{Bmatrix}$$

$$= a_{i1}\alpha_{1j} + a_{i2}\alpha_{2j} + \ldots\ldots + a_{in}\alpha_{nj}$$

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots\ldots + a_{in}A_{jn} \qquad \ldots(1)$$

Now we know from determinants that in a determinant | A | if the elements of any row are multiplied by their corresponding co-factors and added, the result is | A |, and if elements of any line are multiplied by the co-factors of any parallel line and added, then the result is zero.

Thus, R.H.S. of (1) = 0 if $i \neq j$

= | A | if $i = j$

Thus (i, j)th element of A(adj. A) = 0 ($i \neq j$)

= | A | if $i = j$

This shows that all the diagonal elements of A (adj. A) are each equal to | A | and non-diagonal elements are zero.

$$\text{Hence } A (\text{adj. } A) = \begin{bmatrix} |A| & 0 & 0 & \ldots & 0 \\ 0 & |A| & 0 & \ldots & 0 \\ 0 & 0 & |A| & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} = |A| I_n$$

Similarly,

(i, j)th element in (adj. A) A

$$= [\alpha_{i1}\ \alpha_{i2} \ldots \alpha_{in}] \begin{bmatrix} a_{1j} \\ a_{2j} \\ \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \\ a_{nj} \end{bmatrix}$$

$$= \alpha_{i1}a_{1j} + \alpha_{i2}a_{2j} + \ldots + \alpha_{in}a_{nj}$$

$$= A_{1i}a_{1j} + A_{2i}a_{2j} + \ldots + A_{ni}a_{nj}$$

$$= a_{1j}A_{1i} + a_{2j}A_{2i} + \ldots + a_{nj}A_{ni}$$

$$\left.\begin{matrix} = |A| \\ = 0 \end{matrix}\right\} \begin{matrix} \text{if } i = j \\ \text{if } i \neq j \end{matrix} \qquad \text{(By remark Art. 10.23)}$$

$$\therefore \quad (\text{adj. A})(\text{A}) = |\text{A}|\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 \end{bmatrix} = |\text{A}|\,\text{I}_n$$

Hence A (adj. A) = (adj. A)A = | A | I_n.

Cor. If A is a non-singular matrix of order n, then

$$|\text{adj. A}| = |\text{A}|^{n-1}$$

Proof. $\because$ A (adj. A) = | A | I

Taking determinant of both sides

$$\therefore \quad |\text{A (adj. A)}| = ||\text{A}|\,\text{I}|$$

or $$|\text{A}||\text{adj. A}| = |\text{A}|^n \qquad [\because |\text{AB}| = |\text{A}||\text{B}|, |\text{I}| = 1]$$

But $|\text{A}| \neq 0$, dividing by $|\text{A}|$

$$\therefore \quad |\text{adj. A}| = |\text{A}|^{n-1}.$$

Example 4. *Find the adjoint of matrix*

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

and verify the theorem A(adj. A) = (adj. A)A = | A | I.

(M.D.U. 1980 S ; K.U. 1995 A)

Sol. $$|\text{A}| = \begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

operate Col. 1 – 2 Col. 2 ; Col. 3 – 3 Col. 2

$$= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -3 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & -1 \\ -3 & -3 \end{vmatrix} = -(-3-3) = 6$$

Co-factor of 2, (1, 1)th element $= \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$

Co-factor of 1, (1, 2)th element $= -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -(9-2) = -7$

Co-factor of 3, (1, 3)th element $= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 6 - 1 = 5$

Co-factor of 3, (2, 1)th element $= -\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -(3-6) = 3$

Co-factor of 1, (2, 2)th element $= \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 6 - 3 = 3$

Co-factor of 2, (2, 3)th element $= -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -(4-1) = -3$

Co-factor of 1, (3, 1)th element $= \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1$

Co-factor of 2, (3, 2)th element $= -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -(4-9) = 5$

Co-factor of 3, (3, 3)th element $= (-1)^6\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2 - 3 = -1$

$$\therefore \quad \text{Adj. A} = \begin{bmatrix} -1 & -7 & 5 \\ 3 & 3 & -3 \\ -1 & 5 & -1 \end{bmatrix}' = \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\text{A (adj. A)} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.(-1)+1.(-7)+3.5 & 2.3+1.3+3.(-3) & 2.(-1)+1.5+3.(-1) \\ 3.(-1)+1.(-7)+2.5 & 3.3+1.3+2.(-3) & 3.(-1)+1.5+2.(-1) \\ 1.(-1)+2.(-7)+3.5 & 1.3+2.3+3.(-3) & 1.(-1)+2.5+3.(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |\text{A}|\,\text{I}_3$$

$\therefore$ A (adj. A) = | A | I_3.

Similarly, it can be shown that (adj. A)A = | A | . I_3

Hence the verification.

Example 5. *Prove that adjoint of a unit matrix is a unit matrix.*

Sol. Let I_n be the unit matrix of order n.

(i, j)th element of adj. I_n = Co-factor of (j, i)th element in I_n

= 1 or 0 according as $i = j$ or $i \neq j$

($\because$ In a unit matrix, co-factor of diagonal element is 1 whereas co-factor of non-diagonal element is zero)

Thus adj. $\text{I}_n = \text{I}_n$.

Example 6. *Prove that adj. A′ = (adj. A)′, where A is any square matrix.*

Sol. Let A be any square matrix of order n, then adj. A′ and (adj. A)′ are both square matrices of order n.

(i, j)th element of (adj. A)′

= (j, i)th element of (adj. A)

= the co-factor of (i, j)th element in the matrix A

= the co-factor of (j, i)th element in A′

= (i, j)th element of adj. A′

Hence adj. A′ = (*adj.* A)′.

Example 7. *If A is a symmetric matrix, then prove that adjoint A is also symmetric.*

Sol. Let A be a symmetric matrix, then $A' = A$

$\therefore \quad (\text{adj. } A)' = \text{adj. } A'$ (Prove as in Example 6)

$= \text{adj. } A \qquad [\because \; A' = A]$

$\therefore$ (adj. A) is a symmetric matrix.

EXERCISE 10 (*e*)

1. Define adjoint of a matrix.
2. Calculate the adjoints of the following matrices :

(*i*) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 5 & 3 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 2 & -1 & 5 \\ 1 & -2 & 3 \\ 4 & 1 & 2 \end{bmatrix}$. (*K.U. 1992*)

3. (*a*) Given a triangular matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

find adjoint matrix of A. Is adjoint A also a triangular matrix ?

(*b*) Given a symmetric matrix

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

By finding the adjoint of A, show that adjoint of A is also a symmetric matrix.

4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Verify $A (\text{adj. } A) = (\text{adj. } A)A = |A| I_3$, where $|A|$ = determinant of A. (*K.U. 1975 S, 76*)

5. Find the adjoint matrix of

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and verify that

$$A (\text{adj. } A) = (\text{adj. } A)A = |A| I$$

where I is the identity matrix of order 3. (*M.D.U. 1981 S*)

Answers

2. (*i*) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & -5 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

3. (*a*) adj. $A = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix}$, Yes

(*b*) $\begin{bmatrix} bc-f^2 & fg-ch & hf-bg \\ fg-ch & ca-g^2 & gh-af \\ hf-bg & gh-af & ab-h^2 \end{bmatrix}$

which is also a symmetric matrix.

10.29. Inverse of a Square Matrix

Definition. Let A be an n-square matrix. If there exists an n-square matrix B such that

$$AB = BA = I_n, \text{ then}$$

the matrix A is said to be invertible and the matrix B is called the inverse of the matrix A.

Note 1. From the definition given above, it is very clear that if B is the inverse of A, then A is also the inverse of B.

2. A non-square matrix does not have any inverse.

Theorem. *Inverse of a square matrix, if it exists is unique.* (M.D.U. 1980 S ; K.U. 1980)

Proof. Let A be any n-rowed square invertible matrix.

If possible, let B and C both be inverses of A.

$\therefore \quad AB = BA = I_n \quad$ ...(1)

$[\because$ B is inverse of A$]$

and $\quad AC = CA = I_n \quad$ ...(2)

$[\because$ C is inverse of A$]$

Since A, B, C are all square matrices of the same order n.

$\therefore$ The product CAB is defined and

$CAB = C(AB) = CI_n$ [Using (1)]

$= C$ ...(3)

Again, $\quad CAB = (CA)B = I_nB$ [Using (2)]

$= B$ ...(4)

From (3) and (4), we get

$$B = C$$

Thus, inverse of A is unique.

Note. The inverse of A shall, in general, be denoted by A^{-1}.

10.30. Singular and Non-singular matrix

Def. *A square matrix A is said to be singular or non-singular according as* $|A| = 0$ *or* $|A| \neq 0$.

For example, the matrix $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ is singular and $\begin{bmatrix} 1 & 7 \\ 9 & 2 \end{bmatrix}$ is non-singular.

10.31. Theorem

The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$ (i.e., A is non-singular).

(K.U. 1995 A, 92 ; M.D.U. 1980)

Proof. The condition is necessary. Given that A is invertible (*i.e.*, A possesses inverse), to show that A is non-singular.

$\because$ A possesses inverse.

$\therefore$ Let B be the inverse of A.

$\therefore$ $AB = BA = I_n$ [By def. of inverse]

Taking determinants, we get

$$|AB| = |BA| = |I_n| = 1$$

i.e., $|A| \cdot |B| = 1 \neq 0$

But $|A|$ and $|B|$ are scalars (numbers)

$\therefore$ $|A| \neq 0.$

$\therefore$ A is a non-singular matrix.

The condition is sufficient. Given that A is non-singular. To show that A has inverse.

$\because$ A is non-singular, $\therefore$ $|A| \neq 0$

Consider the matrix $B = \dfrac{\text{adj. } A}{|A|}$

Now $AB = A\left(\dfrac{\text{adj. } A}{|A|}\right) = \dfrac{1}{|A|}(A)(\text{adj. } A)$

$$= \frac{1}{|A|}|A| I_n = I_n$$

Also $BA = \left(\dfrac{\text{adj. } A}{|A|}\right)A = \dfrac{1}{|A|}(\text{adj. } A)(A)$

$$= \frac{1}{|A|}|A| I_n = I_n$$

Thus, $AB = BA = I_n$

$\therefore$ A is invertible and B is its inverse.

$\therefore$ $A^{-1} = B = \dfrac{\text{adj. } A}{|A|}.$

Remember. If A is non-singular, then

$$A^{-1} = \frac{\text{adj. } A}{|A|}.$$

Note. The above theorem gives us one of the methods to compute the inverse of a non-singular matrix. We illustrate the method by examples given below.

Example 1. *Find the inverse of matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $ad - bc \neq 0$. (K.U. 1979 S)

Sol. Given matrix is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \therefore \quad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc \neq 0$$

i.e., A is non-singular, $\therefore$ A possesses inverse

$$\text{Adj. } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}' = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

$[\because |A| = ad - bc]$

Remark. This example gives the inverse of every non-singular 2×2 matrix for different values of the element a, b, c, d.

Example 2. *D is the diagonal matrix*

$$\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

where none of the elements d_1, d_2, d_3, d_4 is zero. Find D^{-1}. **(M.D.U. 1991)**

Sol. $$D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

$$|D| = \begin{vmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix}$$

$= d_1 d_2 d_3 d_4 \neq 0$ $[\because$ none of d_1, d_2, d_3, d_4 is zero$]$

Now co-factor of $d_1 = (-1)^2 \begin{vmatrix} d_2 & 0 & 0 \\ 0 & d_3 & 0 \\ 0 & 0 & d_4 \end{vmatrix} = d_2 d_3 d_4.$

Similarly, co-factors of d_2, d_3, d_4 are respectively $d_1 d_3 d_4$; $d_1 d_2 d_4$; $d_1 d_2 d_3$.

$$\therefore \quad \text{adj. A} = \begin{bmatrix} d_2d_3d_4 & 0 & 0 & 0 \\ 0 & d_1d_3d_4 & 0 & 0 \\ 0 & 0 & d_1d_2d_4 & 0 \\ 0 & 0 & 0 & d_1d_2d_3 \end{bmatrix}$$

$$\text{Now} \quad A^{-1} = \frac{\text{Adj. A}}{|A|} = \frac{\text{Adj. A}}{d_1d_2d_3d_4}$$

$$= \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 & 0 \\ 0 & 0 & \frac{1}{d_3} & 0 \\ 0 & 0 & 0 & \frac{1}{d_4} \end{bmatrix}$$

$$= \begin{bmatrix} d_1^{-1} & 0 & 0 & 0 \\ 0 & d_2^{-1} & 0 & 0 \\ 0 & 0 & d_3^{-1} & 0 \\ 0 & 0 & 0 & d_4^{-1} \end{bmatrix}$$

Hence if $D = \text{diag.}\ [d_1, d_2, d_3, d_4]$, where $d_1d_2d_3d_4 \neq 0$

then $\quad D^{-1} = \text{diag.}\ [d_1^{-1}, d_2^{-1}, d_3^{-1}, d_4^{-1}]$.

Remark. The method is quite general and can be extended to a non-singular diagonal matrix of any order.

Example 3. *Find the inverse of the matrix A given by*

$$A = \begin{bmatrix} 9 & 5 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{bmatrix}. \quad \text{(Type K.U. 1994 A ; M.D.U. 1979 S)}$$

Sol. Here

$$|A| = \begin{vmatrix} 9 & 5 & 6 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{vmatrix}$$

Operate $R_1 - 3.R_3$,

$$\therefore \quad |A| = \begin{vmatrix} 0 & -7 & 0 \\ 7 & -1 & 8 \\ 3 & 4 & 2 \end{vmatrix}$$

Expand by Row 1

$$\therefore \quad |A| = -(-7)\begin{vmatrix} 7 & 8 \\ 3 & 2 \end{vmatrix} = 7(14 - 24) = -70 \neq 0.$$

$\therefore$ A is non-singular and A^{-1} exists

$$\text{adj. A} = \begin{bmatrix} -34 & 10 & 31 \\ 14 & 0 & -21 \\ 46 & -30 & -44 \end{bmatrix}' \quad \text{(Replacing each element by its co-factor)}$$

i.e.

$$\text{adj. A} = \begin{bmatrix} -34 & 14 & 46 \\ 10 & 0 & -30 \\ 31 & -21 & -44 \end{bmatrix}$$

$$\text{Hence} \quad A^{-1} = \frac{\text{adj. A}}{|A|} = \begin{bmatrix} \frac{-34}{-70} & \frac{14}{-70} & \frac{46}{-70} \\ \frac{10}{-70} & \frac{100}{-70} & \frac{-30}{-70} \\ \frac{31}{-70} & \frac{21}{-70} & \frac{-44}{-70} \end{bmatrix}$$

i.e.,

$$A^{-1} = \begin{bmatrix} \frac{17}{35} & \frac{-1}{5} & \frac{-23}{35} \\ \frac{-1}{7} & 0 & \frac{3}{7} \\ \frac{-31}{70} & \frac{3}{10} & \frac{22}{35} \end{bmatrix}$$

10.32. Theorem

***(a)* If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$.** (*K.U. 1976*)

***(b)* If A and B are square matrices of order n, then AB is invertible if and only if A and B are invertible and then**

$$(AB)^{-1} = B^{-1}A^{-1}. \quad (\textit{M.D.U. 1981 S ; K.U. 1989})$$

Proof. *(a)* As A is invertible $\therefore$ A^{-1} exists

and $\quad AA^{-1} = A^{-1}A = I.$

This shows (by def.) that A^{-1} is also invertible and inverse of A^{-1} is A

i.e., $\quad (A^{-1})^{-1} = A.$

(b) $\quad |AB| = |A|.|B|$

$\therefore \quad |AB| \neq 0$, if and only if $|A| \neq 0$ and $|B| \neq 0$

i.e., AB is non-singular, if and only if A and B are both non-singular, which is the same as

AB is invertible if and only if A and B are invertible.

Let A and B be invertible and their inverse be A^{-1} and B^{-1}. All these matrices A, B, A^{-1}, B^{-1} are squared matrices of same order *n*.

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [Associative law]

$= (AI)A^{-1} = AA^{-1} = I$

Again $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB$

$= B^{-1}B = I$

$\therefore$ A^{-1}, B^{-1} and $(AB)^{-1}$ all exist.

Also we know that for every non-singular matrix P,

$$P^{-1} = \frac{\text{adj. } P}{|P|}$$

$$\therefore \quad \text{adj. } P = |P| P^{-1} \qquad \ldots(1)$$

Putting P = AB in (1), we have

$$\begin{aligned}\text{adj. }(AB) &= |AB| (AB)^{-1} \\ &= |A||B| B^{-1} A^{-1} \text{ (Reversal Law)} \\ &= |A||B| \frac{\text{adj. } B}{|B|} \frac{\text{adj. } A}{|A|} \\ &= (\text{adj. } B)(\text{adj. } A).\end{aligned}$$

EXERCISE 10 (*f*)

1. Define the inverse of a matrix, and show that whenever it exists, it is unique.
2. Prove that a square matrix A is invertible if and only if, $|A| \neq 0$, where $|A|$ denotes the determinant of A. (*K.U. 1973*)
3. Prove that the inverse of product of two matrices is equal to the product of their inverses but in reverse order. (*M.D.U. 1981 S*)
4. Calculate the inverse of the following matrices whenever exists :

   (*i*) $\begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ (*M.D.U. 1981*)

   (*iii*) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ (*K.U. 1991 S*)

5. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

   Show that $A^{-1} = A$. (*M.D.U. 1980 S*)
6. Let A and B be invertible square matrices of order *n*. Does $(A + B)^{-1}$ exist ? Justify by giving example.
7. If B is non-singular, prove that
   $$|B^{-1} AB| = |A|$$
   A and B being square matrices of the same order.
8. If A is an *n*-square non-singular matrix, prove that
   $$|\text{adj. } A| = |A|^{n-1}.$$
   [**Hint.** Reproduce Ex. 2 Page 301.]
9. If the non-singular matrix A is symmetric, prove that A^{-1} is also symmetric.
10. If the matrices A and B commute, then A^{-1} and B^{-1} are also commute.

Answers

4. (*i*) $\begin{bmatrix} 7 & 3 & -26 \\ 3 & 1 & -11 \\ -5 & -2 & 19 \end{bmatrix}$ 6. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

(*ii*) $\begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

7. Not necessarily.

ORTHOGONAL AND UNITARY MATRICES

10.34. Orthogonal Matrices

A square matrix A is said to be orthogonal if A′A = AA′ = I.

i.e., if $A' = A^{-1}$

For example, the matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

are orthogonal.

[One can verify AA′ = I in each case.]

Every identity matrix is orthogonal.

10.35. The determinant of an orthogonal matrix is ± 1

For, if A is an orthogonal matrix, then

$$\begin{aligned} & AA' = I \\ \Rightarrow\quad & |AA'| = |I| \\ \Rightarrow\quad & |A|.|A'| = I \qquad (\because\ |AB| = |A|.|B| \text{ and } |I| = 1) \\ \Rightarrow\quad & |A|.|A| = 1 \qquad (\because\ |A'| = |A|) \\ \Rightarrow\quad & |A|^2 = 1 \ \Rightarrow\ |A| = \pm 1. \end{aligned}$$

An orthogonal matrix is said to be proper or improper according as its determinant is 1 or – 1.

Note. (*i*) If A is an orthogonal matrix with $|A| = 1$, then each element of A is equal to its cofactor in $|A|$.

(*ii*) If A is an orthogonal matrix with $|A| = -1$ then each element of A is equal to the negative of its cofactor in $|A|$.

10.36. Theorem

The inverse and transpose of an orthogonal matrix are orthogonal.

Proof. Let A be an orthogonal matrix so that

$$AA' = I = A'A$$

Taking inverses, we have

$$(AA')^{-1} = I^{-1}$$

$\Rightarrow \quad (A')^{-1} . A^{-1} = I$

$\Rightarrow \quad (A^{-1})' . A^{-1} = I$

$\Rightarrow$ A^{-1} is orthogonal.

Also, $\quad AA' = I \Rightarrow (A')'A' = I \Rightarrow A'$ is orthogonal.

10.37. Theorem

The product of two orthogonal matrices of the same order is orthogonal. (K.U. 1991)

Proof. Let A and B be two orthogonal matrices of the same order so that

$$A'A = AA' = I \text{ and } B'B = BB' = I$$

Now, $(AB)'(AB) = (B'A')(AB) = B'(A'A)B$

$= B'IB = B'B = I$

Hence AB is orthogonal.

Example 1. *If A is a real skew-symmetric matrix such that $A^2 + I = O$, then A is orthogonal and is of even order.*

Sol. Since A is real skew-symmetric matrix, we have

$$A' = -A$$

$\Rightarrow \quad AA' = -AA$

$\Rightarrow \quad AA' = -A^2$

$\Rightarrow \quad AA' = I \qquad (\because A^2 + I = O \Rightarrow -A^2 = I)$

$\Rightarrow$ A is orthogonal.

Also, $\quad |AA'| = |A|^2 = 1$

$\Rightarrow \quad |A| \neq 0.$

Since A is skew-symmetric and $|A| \neq 0$.

$\therefore$ A is of even order.

[$\because$ By Ex. 3. Page 284 ; Determinant of a skew-symmetric matrix of **odd** order is always zero].

Example 2. *If A and B are two non-singular matrices of the same order such that AA' = BB', show that there exists an orthogonal matrix P such that A = BP.*

Sol. Since $\quad AA' = BB'$,

A and B must be of the same order.

Let $\quad A = BP$

$\Rightarrow \quad P = B^{-1}A \qquad (\because$ B is non-singular, $\therefore B^{-1}$ exists)

Now, $\quad PP' = (B^{-1}A)(B^{-1}A)'$

$= (B^{-1}A)(A'(B^{-1})')$

$= B^{-1}(AA')(B')^{-1}$

$= B^{-1}(BB')(B')^{-1} \qquad (\because A'A = BB')$

$= (B^{-1}B)((B')(B')^{-1}) = I.I = I$

$\Rightarrow$ P is orthogonal.

Hence, there exists an orthogonal matrix

$P(= B^{-1}A)$ such that $A = BP$.

10.38. Unitary Matrix

A square matrix A is said to be unitary if $A^\theta A = I = AA^\theta$.

i.e., iff $A^\theta = A^{-1}$

For example, $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitrary matrix.

For, $A^\theta A = \frac{1}{2}\begin{bmatrix} 1-i & -1-i \\ -1-i & 1+i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I.$

Note. *If each element of A is real, then $\bar{A} = A$*

$\therefore \quad A^\theta = A'$

$\therefore \quad A^\theta A = I \Rightarrow A'A = I$

$\therefore$ *Unitary matrix over R is an othrogonal matrix.*

10.39. Theorem

(*i*) *The transpose of a unitary matrix is unitary.*

(*ii*) *Conjugate of a unitary matrix is unitary.*

(*iii*) *Conjugate transpose of a unitary matrix is unitary.*

(*iv*) *Inverse of a unitary matrix is unitary.* (K.U. 1989 S, 90 S)

(*v*) *Product of two unitary matrices is unitary.*

(*vi*) *The determinant of a unitary matrix has absolute value 1.* (K.U. 1989 S, 90 S)

Proof. (*i*) Let A be a unitary matrix.

$\therefore \quad A^\theta A = I$

$\Rightarrow \quad (A^\theta A)' = I'$

$\Rightarrow \quad A'(A^\theta)' = I$

$\Rightarrow \quad A'(A')' = I \Rightarrow A'(A')^\theta = I$

$\Rightarrow$ A′ is unitary.

(*ii*) Proof is simple.

(*iii*) Proof is simple.

(*iv*) If A is a unitary matrix, then

$A^\theta A = I$

$\Rightarrow \quad (A^\theta A)^{-1} = I^{-1}$

$\Rightarrow \quad A^{-1}(A^\theta)^{-1} = I$

$\Rightarrow \quad A^{-1}(A^{-1})^\theta = I$

$\Rightarrow$ A^{-1} is unitary.

(*v*) Let A, B be two unitary matrices.

$\therefore \quad A^\theta A = I = AA^\theta$